SIMPLIFIED METHOD OF TRACING RAYS THROUGH ANY OPTICAL SYSTEM

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THROUGH ANY

OPTICAL SYSTEM

OF LENSES, PRISMS, AND MIRRORS

BY

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WITH DIAGRAMS

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PREFACE

THE aim of the present volume is to set forth a method of treating the geometrical optics of any given system *intrinsically*, that is to say, without introducing any artificial scaffolding and, therefore, without any *arbitrary splitting* of the entities involved in the problem of tracing a luminous ray through the several surfaces of a system. The resultant formulae, in vector language, of course, as the only appropriate means of intrinsic expression, will then be free of any unnecessary elements, and therefore, if finally translated into any set of scalar formulae for the practical use of the numerical computer, will contain no superfluous, geometrical or arithmetical, complications.

Our purpose is not to treat the whole subject of geometrical optics, but exclusively, or almost so, that part of it which is called by the short name of "ray tracing." This is notoriously the most laborious part of the computer's patient work, and becomes, without question, a formidable task when he has to deal with skew rays and non-centred systems. The problem thus limited can be put shortly:—Given the ray incident upon any system of lenses, mirrors, and prisms, find the emergent ray.

The advantages of the vectorial method of resolving it must not be judged by the conspicuous shortness of the resultant formulae alone, but also by the simplicity of their deduction, as compared with the usual method, and by the facility of recalling the formulae, or of reconstructing them if forgotten. Again, the nature of the proposed method is such as to make the help of drawings—which, especially in the case of skew rays, become, in the best standard treatises, exceedingly complicated—almost superfluous. This circumstance will be particularly welcome to readers who are not endowed with strong visualising powers. In connection therewith

the often troublesome discrimination of sign will become almost automatic.

As already indicated, the mathematical idiom to be employed to secure these advantages and to prevent the intrusion of foreign and artificial elements is Vector Algebra. The reader whose acquaintance with that natural language of space-relations is but slight need not, on that account, be deterred from studying the methods to be described. For they will involve only the rudiments of vector algebra: namely, the addition and the scalar multiplication of vectors together with an occasional application of the vector product of two vectors. A reader who is entirely ignorant of the subject could, in a few hours, acquire this amount of knowledge from one of the existing treatises on it. For instance, the first twenty pages of the author's *Vectorial Mechanics* (Macmillan, 1913) contain all that is required and even more. It is unnecessary, perhaps, to point out that the rudiments of vector addition and multiplication, thus readily learnt, will be found useful in other connections, such as the study of electro-magnetism and kinematics. It is, however, permissible to urge that British students, above all others, should be more widely acquainted with the language of vectors, so powerful and yet comparatively so little used. For it is their own patrimony. After due acknowledgment has been paid to Grassman it remains true that the vector method originated in this country through the creative genius of the great Hamilton, and was shaped into a practical instrument of investigation by Oliver Heaviside's pioneer work of simplification, seconded in America by the independent and equally successful efforts of J. Willard Gibbs.

In order to avoid digressions in the text of this volume it will be well to give here a few explanations of the symbols to be used and to collect the very meagre number of vector formulae which will be required for our purpose. Vectors will be printed in Clarendon, small or capital type. Thus

$$\mathbf{A}, \mathbf{r}, \text{etc.}$$

will be vectors. Their sizes or absolute values will be denoted by the corresponding Italics, as A, r in the above case. If $r=1$, then $\mathbf{r}$ is called a unit vector. The vector sum of two vectors will be denoted, as usual, by $\mathbf{A}+\mathbf{B}$. The scalar product of two vectors $\mathbf{A}$, $\mathbf{B}$ will be written simply $\mathbf{AB}$, and their vector product,

which will not often be needed, V**AB**. With these symbols the only formulae required will be :

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}\,; \quad \mathbf{AB}=\mathbf{BA}\,; \quad \mathrm{V}\mathbf{AB}=-\mathrm{V}\mathbf{BA},$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}\,; \quad \mathrm{V}\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathrm{V}\mathbf{AB}+\mathrm{V}\mathbf{AC}.$$

Similarly for the product of two sums of any number of vectors. If γ is the angle contained between the two vectors **A**, **B**, then

$$\mathbf{AB}=AB\cos\gamma,$$

and the *size* of the vector product V**AB** is

$$|\mathrm{V}\mathbf{AB}|=AB\sin\gamma.$$

The scalar autoproduct of a vector **A**, or its "square," will be written

$$\mathbf{AA}=\mathbf{A}^2=A^2.$$

If **r** is a unit vector, it is evident that $\mathbf{r}^2=1$. The angle γ between two unit vectors **a**, **b**, is given by

$$\mathbf{ab}=\cos\gamma.$$

The mutual perpendicularity or orthogonality of any two vectors **A**, **B** is equivalent to $\mathbf{AB}=0$, while their parallelism is expressed by $\mathrm{V}\mathbf{AB}=0$. Instead of the latter the form $\mathbf{A}\parallel\mathbf{B}$ will sometimes be employed. The concurrency (or equal sense) of parallels and its reverse will, whenever needed, be expressly stated. Finally, for any **A**, **B**, **C**,

$$\mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC}=\mathbf{B}(\mathbf{CA})-\mathbf{C}(\mathbf{AB}).$$

Scarcely more than what has here been set down will be required for following freely the course of the deductions.

The main subject will, now and then, be illustrated by examples or exercises. These will, for the sake of better discrimination between essentials and accessory matter, be printed in small type.

I gladly take the opportunity of expressing my best thanks to Mr. Frank Twyman, Manager to Messrs. Adam Hilger, Ltd., for encouraging this little work, and to my friend Prof. T. Percy Nunn for reading and revising the diction of the MS. My thanks are also due to the Publishers for the care they have bestowed on the book.

L. SILBERSTEIN.

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CONTENTS

SIMPLIFIED METHOD OF TRACING RAYS THROUGH ANY OPTICAL SYSTEM.

1. The Fundamental Reflection and Refraction Laws.

LET the unit vector $\mathbf{n}$ represent the local *normal* to the surface separating two homogeneous isotropic media whose refractive indices are μ and μ'. Whether $\mathbf{n}$ is drawn towards the first or towards the second medium, is immaterial, as will be seen presently. Let the direction of the incident ray be given by the unit vector $\mathbf{r}$, that of the refracted ray by $\mathbf{r}'$, and that of the reflected one by $\mathbf{r}''$, the latter two being again unit vectors. The plane $\mathbf{r}$, $\mathbf{n}$ is then the *incidence plane*. The angles of incidence i, of refraction i', and of reflection i'', Fig. 1, are given, according to the very definition

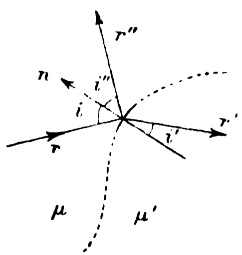

FIG. 1.

of the scalar and the vector product, by

$$\cos i = \mathbf{rn}, \quad \cos i' = \mathbf{r'n}, \quad \cos i'' = \mathbf{r''n},$$

or by the sizes or absolute values of the corresponding vector products, *viz.*

$$\sin i = |\mathrm{V}\mathbf{rn}|,$$

and similarly for the dashed angles. Thus the law of reflection, which asserts the equality of i, i'' and the coplanarity of $\mathbf{r}$, $\mathbf{n}$, $\mathbf{r}''$, can be written, regard being paid to the sense of $\mathbf{r}''$,

$$\mathrm{V}\mathbf{r''n} = \mathrm{V}\mathbf{rn}, \quad \mathbf{r''n} = \mathbf{rn}.*$$

* The second equation, besides being obvious, is also a consequence of the first.

Multiply both sides of the first equation vectorially by $\mathbf{n}$. Then, by the last formula of the Preface, and remembering that $\mathbf{n}^2=1$, the vector form of the *law of reflection* will become

$$\mathbf{r}''=\mathbf{r}-2\mathbf{n}(\mathbf{nr}), \tag{I}$$

giving the reflected ray $\mathbf{r}''$ in terms of the incident ray $\mathbf{r}$ and the incidence normal $\mathbf{n}$.

Again, the refraction law, which asserts that $\mu' \sin i' = \mu \sin i$ and that $\mathbf{r}$, $\mathbf{n}$, $\mathbf{r}'$ are coplanar, is tantamount to writing

$$\mathrm{V}\mathbf{ns}'=\mathrm{V}\mathbf{ns},$$

where $\mathbf{s}$ and $\mathbf{s}'$ stand for $\mu\mathbf{r}$ and $\mu'\mathbf{r}'$, respectively, and might, therefore, be called directed light paths. The name is of small importance. Keep in mind, however, that the $\mathbf{r}$'s being unit vectors, $\mathbf{s}^2=\mu^2$ and $\mathbf{s}'^2=\mu'^2$. Treat the last equation exactly as in the case of reflection, *i.e.* multiply it vectorially by $\mathbf{n}$. Then the result will be

$$\mathbf{s}'=\mathbf{s}+\mathbf{n}(\mathbf{s}'\mathbf{n}-\mathbf{sn}),$$

or, putting

$$g=\mathbf{s}'\mathbf{n}-\mathbf{sn}=\mu'\cos i'-\mu\cos i$$

as a convenient abbreviation, to be used throughout the book,

$$\mathbf{s}'=\mathbf{s}+g\mathbf{n}. \tag{II}$$

This is the required vector form of the *law of refraction*,* giving the refracted ray $\mathbf{s}'$ in terms of the incident $\mathbf{s}$ and the normal $\mathbf{n}$. It is true that the scalar, or ordinary numerical, coefficient g itself contains the unknown vector $\mathbf{s}'$, but only in the combination $\mathbf{s}'\mathbf{n}=\mu'\cos i'$. Now $\mathbf{n}$, $\mathbf{r}$ being given, i is given; whence i', and therefore also g, can be rapidly calculated by means of the relation $\mu'\sin i'=\mu\sin i$. It is, therefore, not advisable to eliminate $\mathbf{s}'$ formally from the right-hand member of (II), although this can easily be done. (*Vide infra*, Section on Multiple Reflection.) It will be found more convenient to leave the form (II) of the refraction law with the numerical coefficient g as it stands.

* Formula (II) has also been arrived at, in a roundabout way, by Sommerfeld and Runge, *Ann. d. Physik*, 35, 1911, p. 277, the object of the authors being to give a general differential basis to geometrical optics. In our case it appears simply as the immediate vector translation of the law of refraction considered as a given law. Cartesian (three formulae) equivalents of (II) are, of course, to be found also in some of the old works. Formula (I) and its consequences were given, in dyadic form, in my paper "On Multiple Reflexion," *Phil. Mag.* for November 1916.

A glance at (I) will suffice to see that the law of reflection follows from that of refraction, (II), by putting in g

$$-\mu \text{ instead of } \mu', \tag{III}$$

a well-known rule. In fact, it is enough for this purpose to multiply (I) on both sides by the refractive index μ of the medium common to $\mathbf{r}$ and $\mathbf{r}''$. Remembering the simple rule (III), we can henceforth use (II) alone as the more general formula. Whenever a separating surface is contemplated as a reflector write $g = -2\mathbf{sn} = -2\mu \cos i$. Unless the contrary is expressly stated we shall conceive every such surface as a refracting one.

Notice that if in (II), and therefore also in (I), $\mathbf{n}$ is replaced by $-\mathbf{n}$, the formulae remain intact.

Let now the refracted ray impinge upon another refracting surface. Then the new refracted ray will be obtained from the ray $\mathbf{s}'$ as this was from $\mathbf{s}$, and so on, for any number of surfaces. Call $\mathbf{n}_1$, $\mathbf{n}_2$, etc., in general $\mathbf{n}_\kappa$, the unit incidence normals of the consecutive surfaces (at the places of incidence of the ray in question) and μ_κ, μ_κ' the indices of the media separated by the κth surface (Fig. 2), so that $\mu_\kappa' = \mu_{\kappa+1}$.

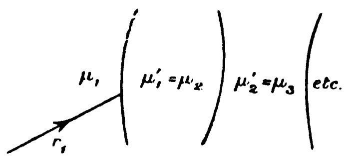

FIG. 2.

Further write, for each surface,

$$\mathbf{s}_\kappa = \mu_\kappa \mathbf{r}_\kappa, \quad \mathbf{s}_\kappa' = \mu_\kappa' \mathbf{r}_\kappa',$$

as above for a single surface, and put $g_\kappa = \mathbf{s}_\kappa'\mathbf{n}_\kappa - \mathbf{s}_\kappa\mathbf{n}_\kappa$. Then, applying the law (II) to the successive surfaces and remembering that $\mathbf{s}_\kappa' = \mathbf{s}_{\kappa+1}$ (newly refracted = actually incident ray), and writing $\mathbf{s}_1 = \mathbf{s}$ for the original incident ray, the finally emergent ray will be given by

$$\mathbf{s}' = \mathbf{s} + \Sigma g_\kappa \mathbf{n}_\kappa, \tag{IV}$$

the sum to be extended over all surfaces.

In the case when all the surfaces are reflectors the last equation assumes a remarkably simple form, as will be seen in a later section, and offers as a final formula conspicuous advantages. In the case, however, of refracting or mixed systems (refracto-reflectors) we shall not actually require the resultant (IV), but shall employ

the formula (II) successively at each surface, after having developed an equally convenient formula for the "transfer" of the data concerning the last refracted ray to the next surface. This latter, *the transfer formula* (which will replace and supersede * the formidable array of what the German opticians call "die Uebergangsformeln"), is an unavoidable supplement of (II), *the refraction formula.* The former will be soon developed in one of the following sections, again without the "help" (which is rather a nuisance or obstacle) of any scaffolding or framework of reference.

2. The Nature of the Data defining an incident Ray.

It will be recalled from the rudiments of vector algebra that all of our vectors, such as **r** or **s**, are free or *non-localized* vectors. That is to say, any line segment of length s and concurrently parallel to a vector **s** drawn concretely somewhere, is, by the very definition of the equality of two vectors, for all purposes of our vector algebra, equal to **s**. In other words, to give **s** means: to give μ and the *direction* of the ray only. Now, if the refracting surface is a plane, this is all that is needed to find the refracted ray $\mathbf{s}'$ (μ', of course, being given). For the whole plane † has but one normal **n**. Thus, unless we desire to know, for instance, the glass thickness traversed by a ray in a prism, the localization of the incident ray **s** is immaterial and not needed. Not so, however, when the refracting interface of the two media is a curved, say a spherical, surface. For then **n** is a function of position on the surface. Thus, **s** giving the direction of the incident ray, we require further data for its localization. Now such a datum is obviously *the place* on the surface where it is struck by the ray **s**. This amounts to two scalar data defining position on the surface, let us say u, v, measured along any network of curvilinear coordinate axes spread out over the surface, as for instance, its lines of curvature. We need by no means specify these u, v. Clearly, "to give the incident ray" means to give **s** ‡ *and* the place (u, v) of its incidence upon the

* In the sense that it will be seen to be applicable to cases to which the usual formulae are not.

† Euclidean space being, of course, tacitly assumed throughout.

‡ Any *two* scalar data, the setter of the problem prefers. We do not count μ involved in $\mathbf{s}=\mu\mathbf{r}$. In general we shall not count data describing the optical system, *viz.* refractive indices, curvatures, and so on.

surface in question. But this being given, the normal $\mathbf{n}$ is also given, which—being a unit vector—amounts precisely to two scalar data.

Thus, whatever the nature of the numerical data for the incident ray, be they angles or projections upon any lines or planes, or distances from axes (artificial or intrinsic), we are fully entitled to say that " incident ray given " means, for a surface σ_κ,

$$\mathbf{s}_\kappa \textit{ and } \mathbf{n}_\kappa \textit{ given.}$$

Such being the case, formula (II) *finds* the refracted ray $\mathbf{s}_\kappa'$. Moreover, it finds it together with its localization ; for the latter ray starts from the point of the surface σ_κ struck by the incident ray. But in order to apply (II) to the next surface $\sigma_{\kappa+1}$ we must transform the last acquired data $\quad \mathbf{s}_\kappa' = \mathbf{s}_{\kappa+1}, \quad \mathbf{n}$

into $\quad \mathbf{s}_{\kappa+1}, \quad \mathbf{n}_{\kappa+1},$

that is to say, we must learn how to replace $\mathbf{n}_\kappa$ by $\mathbf{n}_{\kappa+1}$. It is this which will be given by the transfer formula. Manifestly, having once obtained $\mathbf{s}_{\kappa+1}$, $\mathbf{n}_{\kappa+1}$, we stand before the surface $\sigma_{\kappa+1}$ precisely in the same position and with the same equipment as before the preceding surface. Thus the process of tracing the ray can be continued indefinitely.

There is no essential difficulty in developing, in correspondingly general terms, the transfer formula for a pair of surfaces of any form. But for all purposes actually met with in technical optics, it will be enough to develop the formula for spherical surfaces (of any finite radii), and later on (in a special section on prismatic systems) also for the sub-case of two planes or a plane and a sphere. In the more important case of spheres proper the transfer formula is indispensable for finding even only the direction of the refracted ray, $\mathbf{s}'_{\kappa+1}$, while in the case of planes (prisms) the ray can be traced through the whole system, as far as its direction is concerned, without the " transfer formula " ; the latter being required only when one is interested in the thickness of the several media traversed.

3. The Transfer Formula for Spherical Surfaces.

It was said in what precedes that, the point (u, v) on a surface being given, the incidence normal $\mathbf{n}$ is thereby given. In general, however, the converse manifestly does not hold ; two or more or,

as in the case of a plane surface, even ∞^2 points having parallel normals. But if the refracting surface is a sphere, of centre O, then we have a *one-to-one correspondence* of points (u, v) and normals $\mathbf{n}$, there being no two points to which the same $\mathbf{n}$ belongs. This offers the advantageous possibility of considering the incident ray to be given directly by $\mathbf{s}$ and $\mathbf{n}$. It will be convenient in the case of any spherical surface to draw all the normals $\mathbf{n}$ as unit vectors *from* the sphere's centre as their common origin. The two vectors $\mathbf{s}$, $\mathbf{n}$ being given, we know already, by (II), how to trace the ray through the corresponding surface, *i.e.* how to find the refracted ray $\mathbf{s}'$.

What we still require is to make this ray ready for tracing it through the next surface of the system. This is an easy geometrical problem, although the optical authorities have made it a complicated one by treating it with too many "auxiliaries." Treating it (according to our principle) intrinsically, it is, and will remain, a simple problem.

Let σ_1, σ_2, as representatives of any pair σ_κ, $\sigma_{\kappa+1}$, be two consecutive refracting spherical surfaces of the given optical system. Let O_1, O_2 be their centres, and R_1, R_2 their radii. Draw all the unit normals $\mathbf{n}_1$ *from* O_1, and all $\mathbf{n}_2$ *from* O_2, as a rule. The optical system, as always, being given, so are O_1, O_2, and all the following centres, O_3, etc., collinear or not (centred or not centred system). Thus let D be the scalar distance of the two centres contemplated and $\mathbf{a}$ an unit vector drawn from O_1 towards O_2, as in Fig. 3. (The

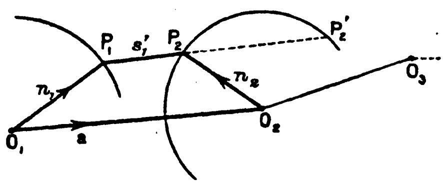

FIG. 3.

reader should note that the surfaces need not turn their convex sides towards one another, nor need O_1 precede O_2; the formulae to be given presently will hold always.) Let $\mathbf{n}_1$ be the particular normal pointing to P_1, the place struck by the incident ray $\mathbf{s}_1$ which penetrates into the second medium as $\mathbf{s}_1' = \mathbf{s}_2$, and strikes the next surface in P_2, a point to which the particular normal $\mathbf{n}_2$

corresponds. Having found $\mathbf{s}_1{}' = \mathbf{s}_2$ by (II), we wish to find P_2, *i.e.* the normal $\mathbf{n}_2$.

In order to obtain it walk round the quadrilateral $O_1P_1P_2O_2$ back to the starting point O_1. This being a closed polygon, no matter whether plane or not, or whether you recross your path * or not, the vector sum of the four vectors traversed is *nil*, *i.e.*, calling L the (scalar) length P_1P_2, or medium "thickness" traversed by the ray, and remembering that $\mathbf{s}_1{}' = \mathbf{s}_2 = \mu_2\mathbf{r}_2$, and therefore, $\overrightarrow{P_1P_2} = L\mathbf{r}_2$,

$$R_2\mathbf{n}_2 = R_1\mathbf{n}_1 - D\mathbf{a} + L\mathbf{r}_2. \tag{1}$$

This, in fact, is already the required transfer formula, expressing the unknown $\mathbf{n}_2$ in terms of things given—all but the scalar coefficient L, a length. In order to find this (which, by the way, is not only needed for "tracing" the ray but will serve the computer in his further task of calculating "astigmatism and curvature"), eliminate from (1) the principal unknown $\mathbf{n}_2$ by squaring both sides of this vector equation. In doing so remember that $\mathbf{n}_2{}^2 = 1$, as are also the squares of the remaining three Clarendons. Then the result will be

$$L^2 - 2(\mathbf{P}\mathbf{r}_2)L = R_2{}^2 - P^2, \tag{2}$$

a quadratic equation for L, with

$$\mathbf{P} = D\mathbf{a} - R_1\mathbf{n}_1 = \mathbf{D} - \mathbf{R}_1, \tag{3}$$

which is a given vector. This finds L and completes the required *transfer formula* (1) which can be written, more shortly,

$$R_2\mathbf{n}_2 = L\mathbf{r}_2 - \mathbf{P}. \tag{1'}$$

Is not this simple? In order to estimate the simplification of the deduction, due to the vector method, read the same subject, for instance, in the well-known collective work of the scientific Jena staff,† where it covers ten pages of print (in which, to be just, not one but three alternative "Uebergangs"-methods are described, none inferior to the others either in rigour or in clumsiness). In harmony with this, their "refraction-formulae" are equally complicated, the whole apparatus consisting of something like fifteen or sixteen formulae. The complication in the scalar

* Which would happen with the choice of another exemplifying figure.

† *Die Bilderzeugung in optischen Instrumenten*, edited by M. von Rohr, Berlin, J. Springer, 1904, pp. 52-62. It is, nevertheless, a valuable work.

treatment is of course due to the circumstance that skew rays are aimed at, not cutting O_1O_2. But as the reader will have noticed, we have nowhere assumed that the incident ray has any such privileged orientation. To repeat, the quadrilateral $O_1P_1P_2O_2$ need not be plane; the resulting formulae (1) or (1′), and (2) with (3) are perfectly general.

The solution of the quadratic equation (2) will, in numerical cases, cost but little trouble, and this will be repaid by obtaining L not only as an auxiliary for the transfer formula, but also as one of the "elements" needed for estimating astigmatic and associated defects of the optical system in question. The equation (2) has, of course, in general *two* roots,

$$L = (\mathbf{Pr}_2) \pm \sqrt{(\mathbf{Pr}_2)^2 + R_2^2 - P^2} \tag{4}$$

corresponding to the two points, P_2 and P_2' (of Fig. 3), in which the ray cuts the second sphere, and in practice there will be no difficulty in discriminating between the root or point (P_2 in the case of Fig. 3) to be retained and that to be rejected. And if both roots coincide or are complex (conjugate), there is an end to the ray-tracing; for in these cases the ray does not penetrate into the next medium.

It would be entirely superfluous to enter at this stage into the details depending on the configuration of O_1, O_2, and the following centres, together with the sense of the curvatures (concave, convex; convex, convex, and so on). The reader will do well to draw several figures for the possible cases, especially for those in which the sides of the quadrilateral cross one another,—and to inspect them carefully. Among other things he will then convince himself that, with the vector $\mathbf{a}$ drawn always from O_1 to O_2 (whether O_1 precedes O_2 or follows it), formula (1), supplemented by (2), (3), remains always valid.

Having obtained $\mathbf{n}_2$ by means of the above formulae, we can proceed to trace the ray through the surface having its centre at O_2; similarly we shall pass from that to the surface centred at O_3, and so forth. No complications will be met with if it should happen that the system is *not centred*, *i.e.* if O_1, O_2, O_3, O_4, etc., are not collinear, or even not coplanar. For the first two surfaces, $\mathbf{a} = \mathbf{a}_{12}$ will be the "axial" unit vector; for the second and third another unit vector $\mathbf{a}_{23}$, drawn from O_2 towards O_3, will replace it, and so on. These vectors, being not artificial but intrinsic,

inherent in the optical system, cannot give rise to any superfluous complications of the calculator's work. If the centres happen to be arrayed in a straight line, all the better : in that case a single unit vector $\mathbf{a}$ will perform the function of *the* axis throughout the tracing process. The calculations will become correspondingly simpler, again for an intrinsic reason, to wit, because the optical system as such is a particularly simple sub-case of the general non-centred system.

4. Summary of the Tracing Process, for Systems of Spherical Surfaces.

It will be well to summarize the whole procedure of tracing a ray through any such system.

The incident ray being given in direction and position, the data of the problem are, in whatever form,

$$\mathbf{s}_1 \quad \text{and} \quad \mathbf{n}_1.$$

From these, $\mu_1 \cos i_1 = \mathbf{s}_1\mathbf{n}_1$ and i_1' by $\mu_1' \sin i_1' = \mu_1 \sin i$, follow immediately, whence the scalar coefficient

$$g_1 = \mu_1' \cos i_1' - \mu_1 \cos i_1.^*$$

This being found we have, by the refraction formula (II),

$$\mathbf{s}_1' = \mathbf{s}_1 + g_1\mathbf{n}_1,$$

which is also $\mathbf{s}_2$, the ray incident upon the second surface of the system.

The vector drawn from O_1 to the next centre O_2 being $\mathbf{D}_{12} = D_{12}\mathbf{a}_{12}$, and

$$\mathbf{P} = \mathbf{D}_{12} - R_1\mathbf{n}_1,$$

the medium thickness L is calculated as the appropriate root

$$L = (\mathbf{P}\mathbf{r}_2) \pm \sqrt{(\mathbf{P}\mathbf{r}_2)^2 + R_2^2 - P^2}$$

of the quadratic equation (2), with $\mathbf{r}_2 = \mathbf{s}_2/\mu_2$.

Finally, by the transfer formula (1) or (1′),

$$\mathbf{n}_2 = \frac{1}{R_2}(L\mathbf{r}_2 - \mathbf{P}).$$

Thus the two vectors are found,

$$\mathbf{s}_2 \quad \text{and} \quad \mathbf{n}_2,$$

ready to trace the ray, thus localized, through the second and

* Which is to be replaced by $-2\mu_1 \cos i_1$, when the surface is contemplated as a reflector.

to carry it over to the third surface, and so forth, up to the last surface of the system, giving the finally emergent ray through two vectors, say,

$$\mathbf{s}' \text{ and } \mathbf{n}',$$

the first determining its direction, and the second localizing it completely with respect to the system.

The results, at any intermediate surface σ_κ, as well as at the last surface of the system, are all of the same form as the original data $(\mathbf{s}_1, \mathbf{n}_1)$.

Each of these pairs, whether of data or results, amounts to the knowledge of four scalar numbers, two involved in $\mathbf{s}$ (not counting μ) and two in the unit vector $\mathbf{n}$.

This exhausts the matter in its general aspect, the procedure being valid for any orientation and position of the incident ray and for any system of spherical surfaces, centred or not.

With regard to its particular or practical aspect, the successful application of this compact set of vector formulae to numerical cases will depend only upon one negative condition, which is: not to spoil their intrinsic simplicity by inappropriate splitting or decomposition of the vectors. A large portion of the remainder of the present work will be dedicated to an attempt to show the reader how to avoid this danger.

As a matter of fact, readers skilled in the handling of vectors will, after what has been said, hardly require any further explanations. This little book, however, being chiefly intended for those who are not experts in the use of vectors, it has seemed indispensable to add some hints at least which are likely to help the majority of readers in applying the general formulae to more or less concrete cases. These will be given in some of the following sections.

5. Hints and Examples.

Having hitherto, in developing the fundamental and general part of the subject, followed a systematic course, we can henceforth without danger of confusion depart therefrom and choose our subjects freely. Our aim being to explain the general procedure by illustrations and hints, a pedantic observance of order would only hamper us unnecessarily.

Let us start with the simple case of a lens consisting of a single piece of glass of index μ limited by two spherical surfaces σ_1, σ_2,

the surrounding medium being air or empty space. To fix our ideas let us take a

Biconcave Lens. Let, as before, O_1, R_1 be the centre and the radius of the first surface, upon which the incident ray $\mathbf{s}=\mathbf{r}$ impinges, and O_2, R_2 the centre and the radius of σ_2, the back surface; finally let $\mathbf{n}_1$, $\mathbf{n}_2$ be the unit normals, drawn from the centres towards the points at which the ray enters and emerges, as in Fig. 4, which however is scarcely needed. By assumption,

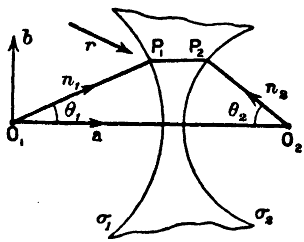

FIG. 4.

the two unit vectors $\mathbf{r}$, $\mathbf{n}$ are given, amounting each to two numerical data. The first practical question is: How are they given? Generally speaking, this will depend upon the decision of the person who sets the problem. Yet, since we are here setting our own problems, let us put the data in a reasonable form, seeking only to avoid artificial frameworks of reference. Now, the lens, consisting but of two spherical surfaces, offers us but one intrinsic directed element, *viz.* its *axis*, which we will represent by the unit vector $\mathbf{a}$ drawn from O_1 towards O_2,—these points themselves having already been exploited by taking them as the origins of the normals. Besides $\mathbf{a}$ there is nothing intrinsic to build with a framework.* Now, if the ray $\mathbf{r}$ happens to lie in a meridian plane (passing through the axis), then nothing more than $\mathbf{a}$ is needed for the ray's relevant definition; say the angles $\mathbf{r}$ and $\mathbf{n}_1$ make with the axis. But let $\mathbf{r}$ be a *skew ray*, not contained in any meridian plane.† Then the lens itself offers us manifestly nothing to complete the determination of the direction $\mathbf{r}$. But the ray

* A plane normal to $\mathbf{a}$ may be a welcome auxiliary, but gives, manifestly, nothing independent of $\mathbf{a}$ itself. Nor shall we look for the rim of the lens to help us.

† The simpler case of a meridian or non-skew ray may be taken up as a sub-case, and as an easy but useful exercise.

itself, as localized, does offer us something, and exactly as much as is needed. In fact, it strikes the first surface in a *given* point P_1, given in one way or another. Through that point draw your "first meridian" plane, marking it with any unit vector $\mathbf{b}$, say perpendicular to $\mathbf{a}$, and count, from that plane or semiplane, the geographical longitude ϕ, the companion of the polar distance or colatitude $\theta_1 = \sphericalangle \mathbf{a}, \mathbf{n}_1$. Thus θ, ϕ will be a polar system of coordinates with centre O_1. Remember that we have drawn our first meridian plane $\mathbf{a}$, O_1, $\mathbf{b}$, only through the end point P_1 of $\mathbf{r}$, but that the ray is in general *not* contained in that plane.

Now, this is all we want. The vector $\mathbf{a}$ being furnished by the lens, and $\mathbf{b}$ by the incident ray, we have everything which is required to write down numerically (scalarly) the data. In fact, "$\mathbf{r}$ given" means now $\mathbf{ra}$, $\mathbf{rb}$ given, say:

$$\mathbf{ra} = \cos\alpha, \quad \mathbf{rb} = \cos\beta. \tag{5}$$

For symmetry, although it is superfluous, introduce a third unit vector $\mathbf{c} = \mathrm{V}\mathbf{ab}$ (normal to $\mathbf{a}$, $\mathbf{b}$, towards the reader), and write

$$\mathbf{rc} = \cos\gamma, \quad \text{so that } (\mathbf{ra})^2 + (\mathbf{rb})^2 + (\mathbf{rc})^2 = 1. \tag{6}$$

Keep in mind, however, that the first two are by themselves sufficient. This settles the question of $\mathbf{r}$. Next, "$\mathbf{n}_1$ given" now means θ_1, ϕ_1 given, *i.e.*, to be general,

$$\mathbf{n}_1 = \mathbf{a}\cos\theta_1 + (\mathbf{b}\cos\phi_1 + \mathbf{c}\sin\phi_1)\sin\theta_1, \tag{7}$$

and since, by the choice of the first meridian, $\phi_1 = 0$,

$$\mathbf{n}_1 = \mathbf{a}\cos\theta_1 + \mathbf{b}\sin\theta_1. \tag{7a}$$

In fine, the complete numerical data, immediately attackable by the computer, are

$$\theta_1, \quad \phi_1 = 0, \quad \mathbf{ra}, \quad \mathbf{rb},$$

the latter two as in (5), or, if he prefers, with $\mathbf{rc}$ as in (6) instead of $\mathbf{rb}$.

Such being the scalar data, act according to the compact procedure of Section 4. Thus *

$$\begin{aligned} \cos i_1 &= \mathbf{r}_1\mathbf{n}_1 = \cos\theta_1(\mathbf{r}_1\mathbf{a}) + \sin\theta_1(\mathbf{r}_1\mathbf{b}) \\ &= \cos\theta_1 . \cos\alpha + \sin\theta_1\cos\beta, \text{ by (7a), (5);} \end{aligned}$$

whence $\quad \sin i_1' = \dfrac{1}{\mu}\sin i_1 \quad$ and $\quad g_1 = \mu\cos i_1' - \cos i_1,$

* Remembering that in the present case $\mathbf{s}_1 = \mathbf{r}_1 = \mathbf{r}$, $\mu_1 = 1$, $\mu_1' = \mu_2 = \mu$, $\mu_2' = 1$.

ready for the first refracted ray

$$\mu \mathbf{r}_1' = \mathbf{r}_1 + g_1 \mathbf{n}_1 = \mu \mathbf{r}_2. \tag{8}$$

If the reader wishes to split this at the present stage he can do so at once, multiplying (8) scalarly by **a**, **b** or **c**. But, being interested only in the finally emergent ray, let us proceed according to the general prescription of Section 4. To "transfer" the ray $\mathbf{r}_2$, found by (8), to the surface σ_2, write D for the distance O_1O_2, and $\mathbf{P} = D\mathbf{a} - R_1\mathbf{n}_1$, as before; then the scalar product required for L will be

$$\begin{aligned}\mathbf{Pr}_2 &= D(\mathbf{r}_2\mathbf{a}) - R_1(\mathbf{r}_2\mathbf{n}_1)\\ &= \frac{D}{\mu}(\cos a + g_1 \cos \theta_1) - \frac{R_1}{\mu}(\cos i_1 + g_1), \text{ by (8), (5), (7a),}\end{aligned}$$

and, the square of **P**,

$$P^2 = D^2 + R_1^2 - 2DR_1 \cos \theta_1, \text{ by (7a).}$$

This finds the glass thickness traversed, which in the present case is, manifestly, the smaller of the two roots of (2),

$$L = (\mathbf{Pr}_2) - \sqrt{(\mathbf{Pr}_2)^2 - P^2 + R_2^2},$$

which the reader will properly contract by using the last two expressions. The "transfer" is thus completed, giving $\mathbf{n}_2$, and therefore also the point P_2, where the ray strikes the back surface,

$$\mathbf{n}_2 = \frac{1}{R_2}(L\mathbf{r}_2 - \mathbf{P}).$$

To know the numerical coordinates of this point may be of immediate interest. Now, calling θ_2, ϕ_2 its colatitude and its longitude, *with O_2 as centre*, but still using the previous first meridian plane, in short the same unit vector **b**, we have just, as in (7),

$$\mathbf{n}_2 = -\mathbf{a} \cos \theta_2 + (\mathbf{b} \cos \phi_2 + \mathbf{c} \sin \phi_2) \sin \theta_2,$$

i.e. for the required two angular coordinates,

$$\cos \theta_2 = -\mathbf{n}_2\mathbf{a}, \quad \cos \phi_2 \sin \theta_2 = \mathbf{n}_2\mathbf{b}, \quad \sin \phi_2 \sin \theta_2 = \mathbf{n}_2\mathbf{c},$$

or, ultimately,

$$\cos \theta_2 = -\mathbf{a}\mathbf{n}_2; \quad \tan \phi_2 = \frac{\mathbf{c}\mathbf{n}_2}{\mathbf{b}\mathbf{n}_2}. \tag{9}$$

Notice that ϕ_1 being zero, ϕ_2 need not necessarily be so; in other words, $\mathbf{n}_2$ is in general *not* coplanar with **a**, $\mathbf{n}_1$. It may be well to recall here that, for any three vectors,

$$\mathbf{AVBC} = \mathbf{BVCA} = \mathbf{CVAB}$$

is the volume of the parallelepipedon constructed on **A**, **B**, **C** as edges, and that, therefore, the sufficient and necessary condition of coplanarity of $\mathbf{n}_1$, $\mathbf{a}$, $\mathbf{n}_2$ is the vanishing of the (scalar) expression

$$R_2\mathbf{a}V\mathbf{n}_1\mathbf{n}_2 = L\mathbf{a}V\mathbf{n}_1\mathbf{r}_2 - \mathbf{a}V\mathbf{n}_1\mathbf{P},$$

or, since **P** is itself coplanar with $\mathbf{a}$, $\mathbf{n}_1$, of the expression

$$\mu\mathbf{a}V\mathbf{n}_1\mathbf{r}_2 = \mathbf{a}V\mathbf{n}_1\mathbf{r}_1, \text{ by (8).}$$

We thus see that $\phi_2 \neq 0$, *i.e.* that P_2 is outside the meridian plane, unless the incident ray $\mathbf{r}_1$ happens to be in a meridian plane. For any *skew* incident ray we have $\phi_2 \neq 0$.

The coordinates ϕ_2, θ_2 of the leaving point P_2 being thus found by (9), the emergent ray is localized, and it remains only to find its direction $\mathbf{r}_2'$. This is, by the refraction formula,

$$\mathbf{r}_2 = \mu\mathbf{r}_2 + g_2\mathbf{n}_2, \quad g_2 = \cos i_2' - \mu \cos i_2, \tag{10}$$

the latter to be found at once from the known

$$\mathbf{r}_2\mathbf{n}_2 = \frac{1}{R_2}(L - \mathbf{P}\mathbf{r}_2),$$

as was g, a moment ago. This completes the tracing of the ray. For, if α', β' be the angles which the finally emergent ray makes with $\mathbf{a}$, $\mathbf{b}$, we have only to multiply (10) scalarly by $\mathbf{a}$, $\mathbf{b}$, to obtain

$$\cos\alpha' = \mathbf{r}_2'\mathbf{a}, \quad \cos\beta' = \mathbf{r}_2'\mathbf{b}.$$

Thus the original incidence data having been put in terms of

$$\theta_1, \quad \phi_1 = 0, \quad \alpha, \quad \beta,$$

the final answer is obtained in similar language, *viz.*

$$\theta_2, \quad \phi_2, \quad \alpha', \quad \beta'.$$

This essentially exhausts the problem of a simple lens. If instead of a biconcave we have a biconvex, or a " vex-cave " lens, etc., the only modifications needed will concern the sign of $\mathbf{a}$ in the expansion of $\mathbf{n}_2$, and the sign of the square root appearing in the " glass thickness " L traversed. The exhaustive discrimination of all possible cases, which will among other things depend also upon the relation of R_1, R_2, D, may be left as an easy, but useful, exercise for the reader. The central glass thickness is, in the above case, $D-(R_1+R_2)$, and similarly, *mutatis mutandis*, for the remaining cases.

Any further information concerning the emergent ray may, whenever required, be easily deduced from the obtained θ_2, ϕ_2; α', β', or vectorially, from $\mathbf{n}_2$, $\mathbf{r}_2'$. Thus, for instance, to decide whether

the emergent ray meets at all the axis of the lens (at a finite or infinite distance), test the triple product

$$p = -\mathbf{a}V\mathbf{r}_2'\mathbf{n}_2, \tag{11}$$

which is a scalar, or ordinary number. If $p=0$, the emergent ray or its prolongation meets the axis somewhere or is parallel to it; and if $p \neq 0$, then $\mathbf{r}_2'$ is a skew ray. By (10) and (8) we have

$$p = \mathbf{a}V\mathbf{n}_2(\mathbf{r}_1 + g_1\mathbf{n}_1), \text{ and so on.}$$

Let the reader complete the development of p, and discuss the circumstances of its vanishing or not. A useful exercise.

As a further exercise, assume (abstractly or in a numerical example) that $p \neq 0$, so that the emergent ray is a skew ray, and find, vectorially, by using $\mathbf{r}_2'$, $\mathbf{n}_2$, the shortest distance of the emergent ray from the lens axis; also the point of the axis which lies at that distance from the ray.

As another exercise treat the trivial case of $p=0$ and find, again vectorially, the distance, from the back surface, of the point of intersection of the ray $\mathbf{r}_2'$ and the axis.

Construct a handy formula for the angle between the original incident ray and the finally emergent one, remembering that its cosine is $\mathbf{r}_1\mathbf{r}_2'$.

The reader will also find it useful to invent for himself similar problems, resolve them vectorially and evaluate the appropriately scalarized expressions numerically. In this way only will he acquire the needed skill in handling not only vectorial optics but vector investigations in general, which, after all, will be found also to be a comparatively pleasant occupation.

Not confining himself to lenses, let him also treat a single or two spherical mirrors, for which g_κ assumes the simpler form $-2\cos i_\kappa$, as explained before.

In actual numerical ray tracings the principal and the accessorial vector formulae will best be written out (and glanced at whenever necessary) in the margin of the computer's sheet, or on an independent little sheet of paper conveniently placed before his eyes. Now and then it will be found convenient to note a vector result by writing out the figures found for its "components," as for example

$$\mathbf{P} = 15\mathbf{a} - 24{\cdot}75\mathbf{n}_1,$$

having previously decided upon the choice of the length unit. It will be remembered, of course, that every *unit vector is dimensionless*, being a vector divided by its own (scalar) size or representative length. Finally, details concerning the manner of writing and arranging the numbers or their logarithms will best be left to the

individuality of each reader. It is a familiar experience that, within very wide limits, everybody finds *his* best way of arranging the numbers by himself, after he has repeatedly executed a given type of computation. It is by no means desirable to drill away human individuality.

6. Prismatic Systems.

Leaving for the present the spherical separation surfaces, let us turn to plane ones. A series of different optical media separated from one another by *plane surfaces* σ_κ constitutes a prism—or prismatic system. This is of almost equal interest and importance with lens systems. As a matter of fact, any prism system is but the simplest sub-case of what has been treated in the first two sections. Yet, in order not to omit any opportunity of familiarizing the reader with the vector treatment of optical problems, let us enter somewhat into the details of these comparatively simple systems.

A single Prism. Let us begin with the case of but two plane surfaces σ_1, σ_2 limiting (to an extent) a medium, say glass, of refractive index μ. Outside $\mu = 1$. In the first place, let us treat the rays apart from their localization, *i.e.* only as far as their directions are concerned. Each refracting face is then fully characterized by a single unit normal, $\mathbf{n}_1$ for the first, $\mathbf{n}_2$ for the second plane face. Remembering that inversion of the $\mathbf{n}$'s leaves all the fundamental formulae intact, draw *both* normals towards the air, as in Fig. 5, so that the prism angle or refracting angle ϵ is given by

$$\mathbf{n}_1\mathbf{n}_2 = \cos(\pi - \epsilon) = -\cos\epsilon. \tag{12}$$

The question of an "auxiliary" framework now entirely disappears. Unlike the simple lens (or any centred lens system) the prism is liberal enough to furnish us, by its own intrinsic elements, a complete natural reference system. For it has, together with $\mathbf{n}_1$, $\mathbf{n}_2$, a well-determined edge. Take a unit vector $\mathbf{e}$ along the edge, normally to the plane of Fig. 5, and towards the page (or downwards if your reading table is horizontal), so that

$$V\mathbf{n}_1\mathbf{n}_2 = \mathbf{e}\sin(\pi - \epsilon) = \mathbf{e}\sin\epsilon. \tag{13}$$

The angle ϵ being given, of course, this is a simple relation between the three natural reference vectors. Returning to Fig. 5, drop the outline of the prism itself, shift the normals $\mathbf{n}_1$, $\mathbf{n}_2$ to a common origin, and draw instead the unit edge $\mathbf{e}$, as in Fig. 5*a*. The prism

will then be represented by the simple three-vector skeleton, with μ subintended.

Now let any ray $\mathbf{r}$ impinge upon (the "first" face of) the prism thus represented. This given ray may also be drawn from the

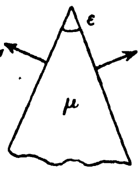

FIG. 5.

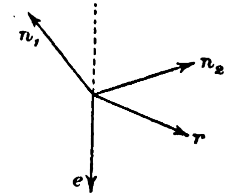

FIG. 5a.

same origin, though it is scarcely necessary. It finally emerges from the prism as $\mathbf{r}_2' = \mathbf{r}'$, say, *viz.* by (IV), Section 1,

$$\mathbf{r}' = \mathbf{r} + g_1\mathbf{n}_1 + g_2\mathbf{n}_2. \tag{14}$$

Here $\mathbf{r}$ is, of course, in general any *skew* ray. The explicit expressions for the scalar coefficients g_1, g_2 can be found at once. But even without the knowledge of their values the vector formula (14) shows us at a glance an interesting and perfectly general property of the prism. Multiply it scalarly by $\mathbf{e}$; then the result will be

$$\mathbf{r}'\mathbf{e} = \mathbf{r}\mathbf{e} = \cos\theta, \text{ say.} \tag{15}$$

Thus the final emergent ray makes with the edge the same angle θ as the original incident ray, whatever its orientation. In other words, the only effect of the prism is to *turn the ray rigidly round the prism edge* as axis. The amount of this rotation depends upon the properties of the prism, μ and ϵ, and upon the orientation of the incident ray.

The ray $\mathbf{r}$ being given, so is the incidence angle $i = \arccos(\mathbf{r}\mathbf{n}_1)$, whence also the coefficient g_1, appearing in the resultant formula (14),

$$g_1 = \sqrt{\mu^2 - \sin^2 i} - \cos i. \tag{16}$$

Similarly the second coefficient g_2 could be written down, as in the case of the lens (and somewhat easier). But it is more elegant to represent fully the final emergent ray $\mathbf{r}'$ without the help of g_2.*

* Especially as the orderly way of calculating g_2 would presuppose the knowledge of the final exit angle $i' = i_2'$.

To get rid of this troublesome coefficient multiply (14) vectorially by $\mathbf{n}_2$; then, by (13), and remembering that $V\mathbf{n}_2\mathbf{n}_2=0$ identically,

$$V\mathbf{r}'\mathbf{n}_2=V\mathbf{r}\mathbf{n}_2+\frac{g_1}{\sin\epsilon}\mathbf{e}. \tag{17}$$

This expresses a vectorial property of the finally emergent ray in terms of things which are all known. Together with the capital property (15) the last equation completely determines the emergent ray in its orientation.

Exercises. Discuss equation (17), deducing from it various vectorial consequences and translating each of these into geometrical language.

Find the expression for the amount of rotation round prism edge as axis, which converts the incident $\mathbf{r}$ into the emergent $\mathbf{r}'$.

Find, in the general case of any skew $\mathbf{r}$, the limit condition, beyond which the ray does not emerge but is internally reflected (at σ_2).

Treat vectorially the trivial, but instructive, case of $\mathbf{r}\perp\mathbf{e}$. (Assume at the outset $\mathbf{r}$ of the form $a\mathbf{n}_1+b\mathbf{n}_2$; a, b scalars.) Translate your results into scalar language, deriving thus the usual formulae.

In practice (spectroscopy, etc.) the aim is, of course, to make $\mathbf{r}\mathbf{e}=0$, or $\mathbf{r}\perp\mathbf{e}$. Yet the manufacturer, and the student in his physical laboratory, can but approach this ideal within, say, $\pm 10''$ or even broader limits. It is, therefore, important to estimate the effect of this error upon the emergent $\mathbf{r}'$. Develop, therefore, the above formulae up to the very end by assuming a value of

$$\mathbf{r}\mathbf{e}=\cos\theta$$

slightly differing from zero, *i.e.* equal to a small fraction. In other words, put $\theta=\frac{\pi}{2}\pm\delta$, where δ is a small angle, say $10''$ or $1'$ or $5'$. Employ, whenever necessary, the well-known developments in series:

$$\left.\begin{aligned}\sin x&=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots,\\ \cos x&=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots.\end{aligned}\right\} \tag{18}$$

Find the effect upon $\mathbf{r}'$ of a small change or "variation" in the direction of the incident ray $\mathbf{r}$. Replace $\mathbf{r}$ by $\mathbf{r}+\delta\mathbf{r}$, remembering that $\mathbf{r}^2=1$, and therefore, $\mathbf{r}\,\delta\mathbf{r}=0$, or $\delta\mathbf{r}\perp\mathbf{r}$, and so on.

Similarly study the effect of a slight variation $\delta\epsilon$ in the refracting angle of the prism, as expressed by (12). Scalarize the resulting formulae and interpret them in physical (geometrical) language.

Medium Thickness traversed. In what precedes, the rays incident upon, transient through and emerging from the prism

were considered with regard to their direction only, independently of their localization. In fact, it was possible for that purpose to replace the prism by a mere triad of unit vectors, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{e}$. The "transfer formula" was not needed owing to the circumstance that the same $\mathbf{n}_2$ served as incidence normal for the whole second plane. This is the reason why the treatment of a prism is conspicuously simpler than that of a lens. In short, the tracing of rays through a prism as far as their *direction* goes requires no "transfer" process. Some device of that kind, however, becomes necessary if we desire to find the length of the ray, whose direction is $\mathbf{r}_1'=\mathbf{r}_2$, within the prism, or, as we will call it for brevity, the medium thickness traversed. Let this be L.

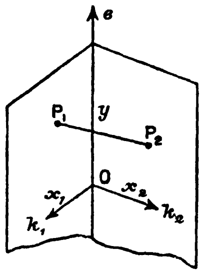

FIG. 6.

Obviously, in order to find it, we must localize the incident ray $\mathbf{r}$ with respect to the prism, by giving, say, the point P_1 where it strikes the first face σ_1. Since P_1 cannot be defined by $\mathbf{n}_1$ (as in the case of a spherical interface), the normal being common to all points of the plane, it is indispensable to determine position on σ_1 by some coordinates. As such we may conveniently take y_1, measured along $\mathbf{e}$ (the edge), from an arbitrary origin O on the edge, and x_1 perpendicularly to the edge. Similarly y_2, x_2 for the second face σ_2 of the prism. We can collect these pairs of coordinates by introducing the vectors

$$\mathbf{p}_1=x_1\mathbf{k}_1+y_1\mathbf{e}, \quad \mathbf{p}_2=x_2\mathbf{k}_2+y_2\mathbf{e}, \tag{19}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$ are two unit vectors along the axes of x_1, x_2, as in Fig. 6.

To obtain L, walk round the triangle OP_1P_2O; then the result will be

$$\mathbf{p}_2=\mathbf{p}_1+L\mathbf{r}_2. \tag{20}$$

This, in the present case, is the "transfer formula," somewhat simpler than in the case of spherical surfaces. Multiply (20) scalarly by $\mathbf{n}_2$, which is normal to $\mathbf{p}_2$; then

$$L\mathbf{r}_2\mathbf{n}_2 = -\mathbf{p}_1\mathbf{n}_2, \tag{21}$$

which finds L, and completes at the same time the prismatic transfer formula. Also, of course, $L^2 = (\mathbf{p}_1 - \mathbf{p}_2)^2$.

Exercises. In (21) substitute $\mathbf{r}_2 = \mathbf{r}_1'$, using the fundamental refraction formula, thus completing the little investigation.

Study L in its dependence upon the prism angle ϵ, and upon the direction and the position of the original incident ray.

Deduce the formulae for the simplest sub-case $\mathbf{re} = 0$.

Find the deviation of the ray, noting that its cosine is $\mathbf{rr}'$. Establish the conditions of minimum deviation.

System of any Number of Prisms. The tracing of any ray, in direction and position, by the successive application of the refraction formula

$$\mathbf{s}_{\kappa+1} = \mathbf{s}_\kappa' = \mathbf{s}_\kappa + g_\kappa \mathbf{n}_\kappa$$

and of the transfer formula typified by (20) with (21), offers no difficulties, nor essentially novel points, and need not, therefore, detain us here. We can imagine that by the application of the said process all the scalar coefficients g_κ are already determined by the given incident ray $\mathbf{r}$ and by the properties of the given system of prisms.

Thus, supposing, for instance, the first and the last medium to be air, the emergent ray $\mathbf{r}'$ will be, by (IV),

$$\mathbf{r}' = \mathbf{r} + \Sigma g_\kappa \mathbf{n}_\kappa = \mathbf{r} + g_1\mathbf{n}_1 + \ldots + g_\kappa \mathbf{n}_\kappa, \tag{22}$$

where $\mathbf{n}_\kappa$ is the unit normal of κth plane face. In this more general case ($\kappa > 2$) it will be convenient to draw all the normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and so on in one and the same sense, say, from the first towards the second, from the second towards the third medium, and so on. Again, as in the case of a single prism; the whole prismatic system can be represented by a skeleton of coinitial unit normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, etc., and unit edges $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, etc. These latter, no matter whether coinciding in space or not, can also be shifted to the same origin. (An air gap will also count as a component "prism," with $\mu = 1$.) Thus a system of $N-1$ prisms, *i.e.* of N interfaces, will be entirely represented by a skeleton of N vectors $\mathbf{n}_\kappa$ and $N-1$ edge vectors $\mathbf{e}_{12}$, etc. If ϵ_1, ϵ_2, etc., are the

prism angles, the relationship of the latter to the former unit vectors will be expressed by

$$\sin \epsilon_1 \,.\, \mathbf{e}_{12} = V\mathbf{n}_1\mathbf{n}_2, \quad \sin \epsilon_2 \,.\, \mathbf{e}_{23} = V\mathbf{n}_2\mathbf{n}_3, \text{ etc.}$$

Any given incident ray $\mathbf{r}$ impinging upon this system will be converted into $\mathbf{r}'$, as expressed by (22), where the g_κ are assumed to have already been found. (The system itself constitutes, of course, the best and more than complete framework of reference, thus giving no place for artificial "auxiliaries.") All properties of the emergent ray can at once be derived from the vector equation (22). We know already that the sole effect of each prism taken separately is to turn the incident ray round the edge as axis, leaving the angle θ with the edge intact. The effect of the whole system will be the resultant of a number of such rotations round $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, etc., as axes, taken in their proper order, of course,—if, as in general will be the case, the edge vectors do not coincide, *i.e.* if the prism edges are not all parallel to one another.

The optical manufacturer attempts in ordinary circumstances (as in the case of spectroscopes) to make the prism edges parallel, *i.e.* $\mathbf{e}_{12} = \mathbf{e}_{23} = \mathbf{e}_{34}$, etc. But, of course, he can succeed in doing so only approximately. Let us call such an ideal system a *correct* one, and any other, with edges more or less deviating from parallelism, a *defective* prism system.* Now, especially if a system of prisms is defective, formula (22) may be very helpful, because with $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, etc., not coplanar, any of the ordinary Cartesian methods would soon lead to the greatest confusion and would require very careful drawings and possibly three-dimensional models to follow the rays in their tortuous course. The vector formula will therefore be particularly useful if it is required to estimate the effect of, say, small deviations from parallelism upon the emergent ray.

If the system of prisms happens to be (or is considered as) a *correct* one, then all the normals $\mathbf{n}_\kappa$ are coplanar and all the unit edge vectors are fused into one, $\mathbf{e}$, say. Multiply (22) by this vector; then the result will be

$$\mathbf{r}'\mathbf{e} = \mathbf{r}\mathbf{e},$$

i.e. $\theta' = \theta$, precisely as for a single prism, as could be expected. Whatever the orientation of the incident ray $\mathbf{r}$, the prism-system

* Although the reader may well imagine cases in which non-parallelism is a feature deliberately aimed at.

turns it rigidly round the common representative edge as axis, leaving its inclination θ intact. Notice also, that all the normals $\mathbf{n}_\kappa$ being coplanar, the vector sum in (22) is again a vector in their plane. Such a vector can always be represented as the sum of two vectors along two fixed unit vectors in (or parallel to) that plane. Let these be $\mathbf{a}$, $\mathbf{b}$. Then (22) will assume the form

$$\mathbf{r}' = \mathbf{r} + A\mathbf{a} + B\mathbf{b}, \tag{23}$$

where A, B are some scalar functions of the orientation of $\mathbf{r}$, and of the characteristics of the prism system, *i.e.* of the μ_κ and the relative orientation of the $\mathbf{n}_\kappa$ (the latter already including all the angles ϵ_κ). These scalar functions A, B can easily be determined for the case of any concretely prescribed prism system. The resultant deviation (angle) $\psi = \arccos(\mathbf{r}\mathbf{r}')$ follows at once by multiplying (23) scalarly by $\mathbf{r}$, which gives

$$\cos\psi = 1 + A\cos(\mathbf{r}, \mathbf{a}) + B\cos(\mathbf{r}, \mathbf{b}). \tag{24}$$

As a useful exercise the reader may treat fully the case of two glass prisms separated by an air-gap, either as an "ideal" or as a "defective" system.

In the latter case it will be of particular interest to fully develop the formula for the effect of a *small* deviation from edge parallelism.

7. System of Plane Mirrors.

Let again, as in Section 6, each interface σ_κ be *plane*, but let each of them be contemplated as a *reflector*. If the position of rays is disregarded, such a system, which may shortly be called a *multiple reflector*, will be entirely determined by the assemblage, and the order of succession, of the unit normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, etc., drawn, let us say, away from the reflecting faces.

As was already hinted at on p. 3, the resultant formula (IV) assumes for any such multiple reflector a particularly simple form.

In fact, returning to the reflection formula (I), written down at the very beginning, and using now a single dash for every reflected ray, we have, for the first reflected ray,

$$\mathbf{r}_1' = \mathbf{r}_1 - 2\mathbf{n}_1(\mathbf{n}_1\mathbf{r}_1), \tag{25}$$

where $\mathbf{r}_1$ is the (original) incident ray. We already know that this is a particular case of the refraction formula (II), page 2,

viz. with μ' replaced by $-\mu$. But there is this important difference, that while (II) contained in its right-hand member, through g, the "unknown" $\mathbf{s}'$, formula (25) contains in its right-hand member $\mathbf{r}_1$ only (besides the given $\mathbf{n}_1$) and no trace of $\mathbf{r}_1'$. It is true that, as was promised on p. 2, the vector $\mathbf{s}'$ can easily be eliminated from the right-hand member; but in doing so we obtain for the refracted ray a formula which in its structure differs essentially from the reflection formula (25). In fact, squaring (II), and remembering that $\mathbf{s}'^2 = \mu'^2$, $\mathbf{s}^2 = \mu^2$, we have

$$g^2 + 2(\mathbf{sn})g = \mu'^2 - \mu^2,$$

whence g, in terms of $\mathbf{sn} = \mu \cos i$, and by substitution in (II),

$$\mathbf{r}' = \rho\mathbf{r} - \rho\mathbf{n}(\mathbf{nr}) + \mathbf{n}\sqrt{\sigma^2 + \rho^2(\mathbf{nr})^2}, \tag{26}$$

where $\rho = \mu/\mu'$ and $\sigma^2 = 1 - \rho^2$. Here $\mathbf{r}$ only appears in the right-hand member; but the operation to be performed upon $\mathbf{r}$ in order to obtain the refracted ray $\mathbf{r}'$ is * manifestly a *non*-distributive one. That is to say, if (26) is written for the moment

$$\mathbf{r}' = \omega\mathbf{r},$$

ω standing for an operator, then $\omega(\mathbf{A} + \mathbf{B})$ is *not* equal $\omega\mathbf{A} + \omega\mathbf{B}$. It is for this reason that we have said (Section 1) that it is not worth while to eliminate $\mathbf{s}'$ from g. Nor would it give us any advantage to study the complicated and non-distributive operator ω, as defined by (26),—which might be called the "refractor."

On the other hand, a glance at the reflection formula (25) will suffice to show that the operator converting the incident into the reflected ray is *distributive*. This is the reason of the simplicity of multiple (plane) reflection, corresponding to a repeated application of such operators.

In fact, $\mathbf{n}_1(\mathbf{A} + \mathbf{B})$ is $\mathbf{n}_1\mathbf{A} + \mathbf{n}_1\mathbf{B}$, which proves the distributivity of the operator Ω_1, say, involved in (25). That is to say, if we write that equation $\mathbf{r}_1' = \Omega_1\mathbf{r}_1$, we have, for any $\mathbf{A}$, $\mathbf{B}$,

$$\Omega_1(\mathbf{A} + \mathbf{B}) = \Omega_1\mathbf{A} + \Omega_1\mathbf{B}.$$

This is a capital property of the "reflector" Ω, as contrasted with the non-distributive property of the "refractor" ω.

In order to see the structure of the "reflector," belonging to the mirror whose normal is $\mathbf{n}_1$, write (25), using the dot (instead of brackets) as separator,

$$\mathbf{r}_1' = \mathbf{r}_1 - 2\mathbf{n}_1 \,.\, \mathbf{n}_1\mathbf{r}_1.$$

* Quite apart from being more complicate than the reflecting operation.

This is but a slight change in notation. Next, instead of writing twice the same operand $\mathbf{r}_1$, put

$$\mathbf{r}_1' = [1 - 2\mathbf{n}_1 . \mathbf{n}_1]\mathbf{r}_1,$$

to mean, by convention, precisely the same thing as before. Thus, the operator Ω_1, converting any incident ray into the corresponding reflected ray, is seen to be

$$\Omega_1 = 1 - 2\mathbf{n}_1 . \mathbf{n}_1, \qquad (27)$$

$\mathbf{n}_1$ being the mirror's unit normal. Such an operator is called a *dyadic* ($\mathbf{n}_1 . \mathbf{n}_1$ being called a dyad, since it contains *two* vectors separated by a dot). The reader need not, however, be deterred by new names or new concepts.* Since he knows how Ω_1 was introduced, there can be no mystery about the meaning or the properties of this operator. In fact, he can read the definition (27) in plain English thus: to operate with Ω_1 upon any given vector $\mathbf{A}$ means to cut off twice its projection upon $\mathbf{n}_1$ along $\mathbf{n}_1$ and to subtract the vector thus obtained from $1 . \mathbf{A}$, that is, from $\mathbf{A}$ itself. This is a very simple operation indeed, more simple than most of those the reader daily performs, whether intellectually or physically. There will thus be no danger of a misunderstanding in using this operator, which is but a very particular example of the powerful dyadic.

Thus our first reflected ray will be

$$\mathbf{r}_1' = \Omega_1 \mathbf{r}_1. \qquad (28)$$

Let now the reflected ray $\mathbf{r}_1'$ impinge upon the second mirror of the system. Then, $\Omega_2 = 1 - 2\mathbf{n}_2 . \mathbf{n}_2$ being the "reflector" of this mirror, the second reflected ray will be, exactly as before,

$$\mathbf{r}_2' = \Omega_2 \mathbf{r}_2 = \Omega_2 \mathbf{r}_1',$$

and substituting $\mathbf{r}_1'$ from (28),

$$\mathbf{r}_2' = \Omega_2 \Omega_1 \mathbf{r}_1, \qquad (28a)$$

which means: operate with Ω_1 upon $\mathbf{r}_1$, then with Ω_2 upon the result of the first operation. This is in essence the same thing as to multiply a given number x first by 3 and then the result by 5. Both kinds are intelligible and reasonably simple operations. The only important difference between them is that while the *order* of the operations in the latter case is a matter of indifference, it

* I mean, new to the reader. The name and the concepts, due to Gibbs, are old enough.

is not so in the former case. $\Omega_2\Omega_1$ is (in general) not the same operation as $\Omega_1\Omega_2$. So also is the actual reflection from mirror 1. followed by a reflection from mirror 2. not the same thing as a double reflection in the reversed order (unless the two mirrors are in a particular relation to one another).

By (28*a*) it can now be said that the "reflector" of a double mirror $(\mathbf{n}_1, \mathbf{n}_2)$ is

$$\Omega_2\Omega_1.$$

Similarly for three, four and any number of plane mirrors. The "reflector" of a *multiple mirror*, consisting of N plane mirrors, acting in the order 1, 2, ... κ, is

$$\Omega = \Omega_N\Omega_{N-1} \ldots \Omega_3\Omega_2\Omega_1,$$

or, more shortly,

$$\Omega = \Pi\Omega_\kappa, \quad \text{where } \Omega_\kappa = 1 - 2\mathbf{n}_\kappa \,.\, \mathbf{n}_\kappa, \tag{29}$$

the "product" extending over all the component mirrors in their prescribed order. If any ray $\mathbf{r}$ impinges upon the multiple mirror, then the final ray, reflected in the said order, is

$$\mathbf{r}' = \Omega\mathbf{r}. \tag{30}$$

The problem is thus reduced to building up the resultant operator Ω from the component ones, Ω_κ. Now, this is done almost as simply as the algebraic multiplication of a succession of polynomials. For to any succession of operators Ω_κ belongs the *associative* property, *i.e.*

$$\Omega_3(\Omega_2\Omega_1) = (\Omega_3\Omega_2)\Omega_1,$$

and similarly for any number of component reflectors.* The said simplicity of handling reflectors Ω_κ and of building them up into the resultant reflector Ω for the multiple mirror is based upon these two properties, the distributivity and the associativity of these operators. In each case the process will consist in simply "multiplying" out the dyads $\mathbf{n} \,.\, \mathbf{n}$ contained in the component reflectors. The only precaution needed is to keep the *order* intact and to remember that juxtaposed vectors not separated by a dot are fused into ordinary scalar products. Thus, for example,

$$\mathbf{n}_1 \,.\, \mathbf{n}_1\mathbf{n}_2 \,.\, \mathbf{n}_2 = \mathbf{n}_1(\mathbf{n}_1\mathbf{n}_2) \,.\, \mathbf{n}_2 = a_{12}\mathbf{n}_1 \,.\, \mathbf{n}_2,$$

where $a_{12} = \mathbf{n}_1\mathbf{n}_2 = \cos(\mathbf{n}_1, \mathbf{n}_2)$ is an ordinary scalar number. Similarly

* Henceforth "reflector" will be used for the operator and "mirror" for the reflecting surface to which it belongs.

for the product of any number of dyads. Such a " product " will always be again a dyad, including an ordinary scalar factor.

In all the following developments we shall employ the short notation

$$\mathbf{n}_\iota \mathbf{n}_\kappa = \cos(\mathbf{n}_\iota, \mathbf{n}_\kappa) = a_{\iota\kappa}, \tag{31}$$

so that, obviously,

$$a_{\iota\kappa} = a_{\kappa\iota}.$$

Before passing on to examples, notice that equation (25) or (28), when squared, gives

$$r_1'^2 = r_1^2 + 4(\mathbf{n}_1\mathbf{r}_1)^2 - 4(\mathbf{n}_1\mathbf{r}_1)^2 = r_1^2.$$

Thus, the incident $\mathbf{r}_1$ being a unit vector, so is also the reflected $\mathbf{r}_1'$. The same property holds, of course, for any multiple mirror, *i.e.* if $\mathbf{r}' = \Omega\mathbf{r}$, as in (30), then $r'^2 = r^2 = 1$. Notice also that the iteration of any Ω_κ is 1 or an *idem factor*; in fact,

$$\Omega_\kappa\Omega_\kappa = 1 - 4\mathbf{n}_\kappa \, . \, \mathbf{n}_\kappa + 4(\mathbf{n}_\kappa \, . \, \mathbf{n}_\kappa)^2 = 1 - 4\mathbf{n}_\kappa \, . \, \mathbf{n}_\kappa + \mathbf{n}_\kappa \, . \, \mathbf{n}_\kappa$$

(since $\mathbf{n}_\kappa^2 = 1$), whence $\Omega_\kappa\Omega_\kappa$ or $\Omega_\kappa^2 = 1$, which is an obvious property.

If ψ be the angle contained between the incident and the finally reflected ray, we have

$$\cos\psi = \mathbf{r}\Omega\mathbf{r}, \tag{32}$$

for any multiple mirror. The expression on the right hand is the scalar product of $\mathbf{r}$ into the vector $\Omega\mathbf{r}$, or, which turns out to be precisely the same thing, of the vector $\mathbf{r}\Omega$ into $\mathbf{r}$. Ω being a dyadic, composed of dyads $\mathbf{a} \, . \, \mathbf{b}$, can manifestly be used either as a postfactor or a prefactor. For any of the component dyads can be used so. Thus

$$\mathbf{a} \, . \, \mathbf{br} \text{ means } \mathbf{a}(\mathbf{br})$$

and

$$\mathbf{ra} \, . \, \mathbf{b} \text{ means } \mathbf{b}(\mathbf{ar}),$$

both being perfectly definite results (vectors), in general differing from one another, of course. But $\mathbf{r}(\Omega\mathbf{r})$ is the same thing as $(\mathbf{r}\Omega)\mathbf{r}$, so that in (32) and all similar equations no brackets are needed.

The reflector Ω belonging to any N-uple mirror will be obtained by multiplying out (with the said precautions) all the component reflectors Ω_κ. Thus, to obtain Ω for a quadruple mirror, say, we must necessarily have first the operators for a triple, and, before that, for a double mirror. Similarly for any N. It is therefore but natural to begin with the double mirror, and, before passing on to the triple and more complicated mirrors, to stop a while at each of the intermediate stages, discussing and interpreting the results. It will be understood that by an " N-uple " not neces-

sarily a system of N different simple mirrors is meant, but only that N successive reflections (in a prescribed order) are contemplated. An incident ray can be reflected from two mirrors, for instance, three or very many times, as is the case, for example, in the plan-parallel piece of a Lummer plate.

Double Mirror. The cosine of the angle between the two component mirrors being $a_{12}=\mathbf{n}_1\mathbf{n}_2$, as in (31), we have, by (29), for the reflector of our double mirror,

$$\Omega=\Omega_2\Omega_1=[1-2\mathbf{n}_2 \,.\, \mathbf{n}_2][1-2\mathbf{n}_1 \,.\, \mathbf{n}_1],$$

i.e., with due regard to the prescriptions of the "multiplication" rule,

$$\Omega=1-2\mathbf{n}_1 \,.\, \mathbf{n}_1-2\mathbf{n}_2 \,.\, \mathbf{n}_2+4a_{12}\mathbf{n}_2 \,.\, \mathbf{n}_1, \tag{33}$$

or, introducing the vector $\mathbf{m}=\mathbf{n}_1-2a_{12}\mathbf{n}_2$,

$$\Omega=1-2\mathbf{m} \,.\, \mathbf{n}_1-2\mathbf{n}_2 \,.\, \mathbf{n}_2. \tag{33a}$$

This, of course, is valid for any operand, that is to say, for any incident ray $\mathbf{r}$, giving $\mathbf{r}'=\Omega\mathbf{r}$ for the reflected ray. Thus

$$\mathbf{r}'=\mathbf{r}-2r_1\mathbf{m}-2r_2\mathbf{n}_2,$$

where $r_1=\mathbf{r}\mathbf{n}_1$, $r_2=\mathbf{r}\mathbf{n}_2$ are the given projections of the incident ray upon the mirror normals. Since the edge $\mathbf{e}$, along which the two mirrors intersect, is perpendicular to $\mathbf{n}_1$, $\mathbf{n}_2$, and therefore also to $\mathbf{m}$, we have again, as in the case of a prism,

$$\mathbf{r}'\mathbf{e}=\mathbf{r}\mathbf{e}=\cos\theta.$$

In fact, the double mirror is but a sub-case of the prism (for μ' replaced by $-\mu$). The angle θ which the ray makes with the edge remains intact after the double reflection. This is a general property of the double mirror independent of the value of a_{12}.

From (33) we see that $\Omega_2\Omega_1$ differs from $\Omega_1\Omega_2$ since the last dyad $\mathbf{n}_2 \,.\, \mathbf{n}_1$ is *not symmetrical* ($\mathbf{n}_2$, $\mathbf{n}_1$ are two different directions). Thus, if a beam of parallel rays $\mathbf{r}$ is broad enough to impinge upon both mirrors, it is split by the double mirror into *two* beams, $\mathbf{r}'$ as before and $\mathbf{r}''=\Omega_1\Omega_2\mathbf{r}$, so that

$$\mathbf{r}'-\mathbf{r}''=4a_{12}[\mathbf{n}_2 \,.\, \mathbf{n}_1-\mathbf{n}_1 \,.\, \mathbf{n}_2]\mathbf{r}. \tag{34}$$

In particular, if the double mirror is *orthogonal*, that is to say, if $a_{12}=\mathbf{n}_1\mathbf{n}_2=0$, we have $\mathbf{r}'=\mathbf{r}''$,

i.e. the two reflected beams coincide. In this case the operator (33) becomes a *symmetrical* (or self-conjugate) dyadic, *viz.*

$$\Omega=\Omega_2\Omega_1=\Omega_1\Omega_2=1-2\mathbf{n}_1 \,.\, \mathbf{n}_1-2\mathbf{n}_2 \,.\, \mathbf{n}_2.$$

Again, since $\mathbf{n}_1 \perp \mathbf{n}_2$, we have

$$\mathbf{n}_1 . \mathbf{n}_1 + \mathbf{n}_2 . \mathbf{n}_2 + \mathbf{e} . \mathbf{e} = 1,$$

for if $\mathbf{i}, \mathbf{j}, \mathbf{k}$, be any normal system of unit vectors (as is $\mathbf{n}_1, \mathbf{n}_2, \mathbf{e}$) we have manifestly $[\mathbf{i} . \mathbf{i} + \mathbf{j} . \mathbf{j} + \mathbf{k} . \mathbf{k}]\mathbf{V} = \mathbf{V}$, for any vector $\mathbf{V}$. Therefore, for an *orthogonal* double mirror, whose unit edge is $\mathbf{e}$,

$$\Omega = -1 + 2\mathbf{e} . \mathbf{e}. \tag{35}$$

That is to say, the reflection from such a mirror is, independently of the order (12 or 21), wholly equivalent to the reflection from a simple mirror whose *normal* is $\mathbf{e}$, followed by a simple reversal (-1). This holds for any incident ray $\mathbf{r}$. If the incident ray happens to be normal to the edge, *i.e.* if $\mathbf{re} = 0$, we have

$$\mathbf{r}' = \Omega\mathbf{r} = -\mathbf{r},$$

that is to say, the ray is sent back parallel to itself. This simple property is also familiar from ordinary geometrical constructions. But none of the cases of a skew $\mathbf{r}$ (oblique to $\mathbf{e}$) will be found so easy to treat by the usual methods, based on a good drawing and spherical trigonometry, as by the proposed vectorial or dyadical method.

Returning once more to the general formula (33), we have for the angle ψ between the incident and the finally reflected ray, by (32),

$$\tfrac{1}{2}(1 - \cos\psi) = r_1^2 + r_2^2 - 2a_{12}r_1r_2;$$

on the other hand, since $\mathbf{r} = r_1\mathbf{n}_1 + r_2\mathbf{n}_2 + (\mathbf{re})\mathbf{e}$ is a unit vector,

$$r_1^2 + r_2^2 + 2a_{12}r_1r_2 = 1 - (\mathbf{re})^2 = \sin^2\theta,$$

so that the last equation can be written

$$\tfrac{1}{2}(1 - \cos\psi) = \sin^2\theta \; = 4a_{12}r_1r_2, \tag{36}$$

giving the angle $\psi = \mathbf{r}, \mathbf{r}'$ for any double mirror and for any given incident ray $\mathbf{r} = r_1\mathbf{n}_1 + r_2\mathbf{n}_2$.

From the general formula (36) we see without difficulty that there is no such double mirror which would send back (parallel to its own path) *every* incident ray; in short, that a double mirror cannot be a "central" mirror, as are those used in modern signalling. In fact, $\cos\psi = -1$ would imply $4a_{12}r_1r_2 + (\mathbf{re})^2 = 0$, and this cannot be satisfied for all directions of $\mathbf{r}$.

Further discussions of the double mirror will be left to the reader as an exercise. In this case the adaptation of the vector formulae to numerical purposes is almost immediate, and calls, therefore, for no further explanations.

Triple Mirror. The resultant reflector in this case is $\Omega = \Omega_3\Omega_2\Omega_1$, the product of $\Omega_3 = 1 - 2\mathbf{n}_3 \cdot \mathbf{n}_3$ into the operator of the preceding case already developed in (33). Thus, for any triple mirror, the order of reflections being 1, 2, 3, the resultant operator $\Omega = \Omega_{123}$ is

$$\Omega_{123} = 1 - 2[\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3] \\ + 4[a_{12}\mathbf{n}_2 \cdot \mathbf{n}_1 + a_{23}\mathbf{n}_3 \cdot \mathbf{n}_2 + a_{31}\mathbf{n}_3 \cdot \mathbf{n}_1] - 8a_{12}a_{23}\mathbf{n}_3 \cdot \mathbf{n}_1. \quad (37)^*$$

Here again, the second term being symmetrical, but not the third, nor the last, Ω_{132}, etc., will differ from Ω_{123}. In other words, an incident beam of parallel rays $\mathbf{r}$ will give rise to *six* reflected beams represented by

$$\Omega_{123}\mathbf{r}, \quad \Omega_{132}\mathbf{r}, \quad \Omega_{231}\mathbf{r}, \text{ etc.},$$

which will, generally speaking, deviate from one another. These six reflected beams become parallel to one another, *i.e.* Ω becomes independent of the order of the three successive reflections when and only when

$$a_{12} = a_{23} = a_{31} = 0,$$

abolishing the third and the fourth terms in (37), *i.e.* when the three component mirrors are mutually *perpendicular*. Moreover, $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ being in that case a triad of normal unit vectors, the dyadic $\mathbf{n}_1 \cdot \mathbf{n}_1 + \mathbf{n}_2 \cdot \mathbf{n}_2 + \mathbf{n}_3 \cdot \mathbf{n}_3$ becomes an idemfactor or 1, and the operator is reduced to

$$\Omega = -1\,; \quad \therefore\ \mathbf{r}' = -\mathbf{r}. \quad (37a)$$

That is to say, every incident ray, independently of the order of its three reflections, is sent back parallel to itself. The *orthogonal* triple mirror is a *central* mirror. Such triple mirrors, in fact, are used in practice, their orthogonality being now attainable to a high degree of approximation.

Returning to the general triple mirror, write r_1, r_2, r_3 for the direction cosines of the incident ray or its projections upon the three normals $\mathbf{n}_\kappa$. Then the direction cosines of the reflected ray $\mathbf{r}'$ will be linear homogeneous functions of r_1, r_2, r_3, *viz.*, by (37),

$$r_1' = [-1 + 4a_{12}^2 + a_{31}^2 - 8a_{12}a_{23}a_{31}]r_1 - [2a_{12} + 4a_{23}a_{13}]r_2 - 2a_{13}r_3, \quad (38)$$

and so on.

* This may, at first sight, seem very complicate. Try, however, to treat the general triple mirror by the ordinary methods. Then (37) will be found very simple.

Exercises. Write down, by the general formula (37), the remaining two direction cosines r_2', r_3', thus obtaining three scalar formulae with $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ as reference system.

Assuming that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are not coplanar, *i.e.* that the three mirrors constitute a proper *pyramid* (not a prism), any incident ray $\mathbf{r}$ can be put into the form

$$\mathbf{r} = p_1\mathbf{n}_1 + p_2\mathbf{n}_2 + p_3\mathbf{n}_3,$$

where p_1, etc., are some scalars. Determine them in terms of the orthogonal projections $r_1 = \mathbf{r}\mathbf{n}_1$, etc. (Notice that $p_1 = r_1$, etc., when and only when the mirrors are mutually perpendicular.)

Similarly determine the coefficients of

$$\mathbf{r}' = p_1'\mathbf{n}_1 + p_2'\mathbf{n}_2 + p_3'\mathbf{n}_3,$$

and replace the formulae (38) by three equations giving the p_κ' in terms of the p_κ, and in terms of the properties of the mirror, *i.e.* a_{23}, etc. Discuss the general formulae thus obtained, and apply them to a number of particular cases.

Treat the prismatic arrangement of the three component mirrors, when $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are coplanar; *i.e.* such that $a\mathbf{n}_1 + b\mathbf{n}_2 + c\mathbf{n}_3 = 0$, a, b, c being three scalars, and so on. Consider first an incident ray $\perp$ to the edges (all representable by one $\mathbf{e}$), and then any oblique or skew ray. In the former case imagine the mirror pierced at a place, just to let through the incident ray; or else let the mirrors consist of non-silvered (plane parallel) glass plates.

The angle ψ between the incident and the finally reflected ray is given by $\cos\psi = \mathbf{r}\Omega\mathbf{r}$, *i.e.*, according to (37), by

$$\tfrac{1}{2}(1 - \cos\psi) = (r_1^2 + r_2^2 + r_3^2) - 2[a_{12}r_1r_2 + \text{etc.}] + 4a_{12}a_{23}r_3r_1. \quad (39)$$

Each of the two bracketed expressions is symmetrical with respect to the suffixes 1, 2, 3, but the last term is not. Thus, if we replace the order of reflections 123, by 132, say, and call ψ' the angle between the corresponding finally reflected ray and $\mathbf{r}$, we have the elegant formula

$$\cos\psi' - \cos\psi = 8r_1a_{23}(a_{12}r_3 - a_{13}r_2). \quad (40)$$

Similarly for any other permutation. Notice that 321 gives the same reflected ray as 123.

If the three mirrors constitute a *regular pyramid, i.e.* if

$$a_{12} = a_{23} = a_{31} = \cos\omega, \text{ say,}$$

then formula (39) becomes

$$\tfrac{1}{2}(1 - \cos\psi) = (r_1^2 + r_2^2 + r_3^2) - 2\cos\omega(r_1r_2 + r_2r_3 + r_3r_1) + 4r_1r_3\cos^2\omega. \quad (39a)$$

More especially, if the incident ray $\mathbf{r}$ is *equally inclined* to the three reflecting planes of the regular pyramid, we have

$$r_1 = r_2 = r_3 = \frac{1}{\sqrt{3}} \cot \frac{\omega}{2},$$

and therefore, remembering that $1 - \cos\psi = 2\sin^2\frac{\psi}{2}$,

$$\sin\frac{\psi}{2} = \cot\frac{\omega}{2}\sqrt{1 - 2\cos\omega + \frac{4}{3}\cos^2\omega}. \tag{39b}$$

In this case the angle ψ is, of course, independent of the order of reflections. The reflected beams (corresponding to a beam of parallel incident rays) are not parallel to one another, but are equally inclined to and symmetrically disposed around the direction of the incident beam. These reflected beams will coincide in direction when and only when $\omega = 90°$, *i.e.* when the mirror becomes an orthogonal and, therefore, a central mirror. In fact, for $\omega = \frac{\pi}{2}$, formula (39*b*) gives $\sin\frac{\psi}{2} = 1$, *i.e.* $\psi = 180°$, which means a reversal of the incident ray. This, as we already know, is the property of an orthogonal mirror with respect to *any* incident ray.

Quadruple and more Complicated Mirrors. Multiplying (37) by $1 - 2\mathbf{n}_4 \,.\, \mathbf{n}_4$, where $\mathbf{n}_4$ is the normal of the fourth mirror, the reader will obtain the operator $\Omega = \Omega_{1234}$, and similarly for quintuple reflections, and so on. As was already remarked, some of the normals $\mathbf{n}_\kappa$ may reappear, that is, we may have, for instance, a quintuple reflection from but two mirrors; if such be the case, it is enough to put $\mathbf{n}_3 = \mathbf{n}_5 = \mathbf{n}_1$, and $\mathbf{n}_4 = \mathbf{n}_2$.

As an example of this kind the reader may treat the rotation and progress of a skew ray incident upon an interior face of an equilateral prism built up of three plane mirrors; $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ coplanar, $\mathbf{n}_4 = \mathbf{n}_1$, $\mathbf{n}_5 = \mathbf{n}_2$, and so on.

Reversal of the Order of Reflections. For any multiple mirror, let $\mathbf{r}$ be the incident ray, $\mathbf{r}'$ the finally reflected ray when the order of reflections is $123 \ldots N-1, N$, and $\mathbf{r}''$ the finally reflected ray when the order of reflections is *reversed*, *i.e.* $N, N-1, \ldots 321$. Then, Ω being $\Omega_N\Omega_{N-1} \ldots \Omega_3\Omega_2\Omega_1$, as on page 25, we have

$$\mathbf{r}' = \Omega\mathbf{r} \quad \text{and} \quad \mathbf{r}'' = \mathbf{r}\Omega,$$

whence

$$\mathbf{r}\mathbf{r}' = \mathbf{r}\Omega\mathbf{r}, \quad \mathbf{r}''\mathbf{r} = \mathbf{r}\mathbf{r}'' = \mathbf{r}\Omega\mathbf{r},$$

and therefore, for any multiple mirror,

$$\mathbf{r}'\mathbf{r} = \mathbf{r}''\mathbf{r}, \tag{41}$$

while $\mathbf{r}'\mathbf{r}''=\mathbf{r}\Omega^2\mathbf{r}$. That is to say, the reflected rays $\mathbf{r}'$, $\mathbf{r}''$, although not parallel to one another, are *always equally inclined to the incident beam* $\mathbf{r}$. The equality $\psi_{123}=\psi_{321}$ mentioned on page 30 is but a special instance of this general property.

8. Dyadic representing the most General Reflector.

Consider the successive reflections of a ray $\mathbf{r}$ from any number of mirrors, whose normals are $\mathbf{n}_1$, $\mathbf{n}_2$, etc., and let Ω, with the order of reflections subintended, be the corresponding resultant operator or the "reflector" belonging to the whole, ordered, system. (This comprises also the case of any number of curved mirrors; the only difference being that when they are plane, all the normals are immediately given, and when curved, the successive normals are to be found by the application of the "transfer formula," developed in a previous Section.) Now Ω being the product of dyadics of the form

$$1-2\mathbf{n}\,.\,\mathbf{n},$$

is itself a dyadic. And since our space has but three dimensions, all the *consequents* * of any reflector, no matter how complicated the system to which it belongs, can be expressed as linear homogeneous functions of *three non-coplanar vectors*, which can always be made *unit* vectors, let us say, $\mathbf{f}, \mathbf{g}, \mathbf{h}$. The antecedents belonging to these consequents will then, for any given multiple mirror, be certain three vectors, let us say, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Thus, the most general reflector will have the form of a *trinomial dyadic*,

$$\Omega=\mathbf{a}\,.\,\mathbf{f}+\mathbf{b}\,.\,\mathbf{g}+\mathbf{c}\,.\,\mathbf{h}. \tag{42}$$

The plain meaning of this is that any incident ray $\mathbf{r}$ will be reflected, in the prescribed order, as

$$\mathbf{r}'=\mathbf{a}(\mathbf{fr})+\mathbf{b}(\mathbf{gr})+\mathbf{c}(\mathbf{hr}). \tag{42a}$$

The consequents $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ need not necessarily constitute a normal system; it may sometimes be convenient to choose an oblique system of vectors. The choice will depend upon the properties of the system of mirrors. •

The above result may also be stated by saying that the most general Ω is a *linear vector operator*, or $\mathbf{r}'$ a linear homogeneous vector function of $\mathbf{r}$, as was already mentioned.

* According to Gibbs' terminology, the first vector, $\mathbf{a}$, in a dyad $\mathbf{a}\,.\,\mathbf{b}$ is called the *antecedent*, and the second vector, $\mathbf{b}$, the *consequent*.

Splitting into Cartesians, for instance, and counting up the coefficients expressing the components of $\mathbf{r}'$ in terms of those of $\mathbf{r}$, the reader will see at once that the most general linear vector operator implies *nine* independent scalar or numerical data. Let us count the number of such data, implied in the most general reflector Ω, upon the formula (42) itself.

The consequents, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, which can always be taken as unit vectors, and, moreover, always as a normal system of unit vectors,* imply $2+1+0=3$ scalar data (for the direction $\mathbf{f}$ being fixed, $\mathbf{g}$ requires but one datum, and $\mathbf{h}$ is codetermined, apart from the irrelevant sign or sense). Now it is always (*i.e.* for *any* trinomial dyadic) possible to give to the normal system $\mathbf{f}, \mathbf{g}, \mathbf{h}$ such an orientation as to convert the antecedents also into some *normal system* of vectors,† but, of course, not unit vectors, after $\mathbf{f}$, etc., have been made so. Thus the orientation of the system of antecedents implies three more data, and the size or absolute value of each antecedent one datum. Thus the total number of scalar data implied in the most general trinomial dyadic is *nine*.

And since our reflector Ω is such that it leaves the size of any operand unaltered, *i.e.*

$$\mathbf{r}'^2 = \mathbf{r}^2$$

for any $\mathbf{r}$, the number of independent scalar coefficients contained in *the most general reflector* Ω is reduced to *eight*. These coefficients depend upon the properties of the system of mirrors and, of course, of the prescribed order of successive reflections. No matter, however, what their arrangement and how many of simple mirrors are built up into the system, all of its properties are crowded into eight coefficients,—corresponding to each prescribed order. On permutating the order, the operator, or its coefficients, will, in general, be changed; in some special cases they may remain intact. The reversal of order has, however, the perfectly general property expressed by (41).

A dyadic, or linear vector operator, such that $\Omega\mathbf{r} = \mathbf{r}\Omega$, for any $\mathbf{r}$, is called a *self-conjugate* or symmetrical one. The simple reflector

* Although this may sometimes be not the most convenient choice. This, however, does not influence the result of our counting.

† A simple proof is given in J. W. Gibbs' lectures on *Vector Analysis*, edited by E. B. Wilson, New York, 1907, p. 304. Our dot, a separator, is not to be confounded with Gibbs' dot, which is the symbol of scalar multiplication. The notation adopted in the present volume is essentially that of Oliver Heaviside.

$1 - 2\mathbf{n}.\mathbf{n}$ is, obviously, such an operator. But that corresponding to the general double mirror is no longer self-conjugate. Other details of the discussion of the multiple reflection formulae may be left to the reader. As to further examples and exercises, besides those proposed, the reader will doubtless invent them himself without difficulty.

9. Differential Properties of Refracted or Reflected Pencils.

Although the chief object of this book, *viz.* the tracing of a single ray, is in its essence exhausted by what has been said in the preceding sections, it may be well to add one more section in order to give the reader at least some cursory hints about the treatment of differential properties of beams or pencils of rays sent through an optical system: properties, that is to say, which concern the passage from one ray of a pencil to its neighbours, or the passage from a given wave-length, and the corresponding index μ, to an infinitesimally different one (astigmatism, chromatic aberration).

Problems of this kind do not, as a matter of fact, require anything essentially different from what has already been given in the earlier part of the book. They are solved, in each case, by simply differentiating both sides of the vector formulae concerning a single ray, obtainable by the above method. Now, the differentiation of vector expressions does not differ from that of scalar ones, provided—of course—that we differentiate with respect to a scalar magnitude as the independent variable or parameter. As a precaution hardly anything more is needed than the preservation of order of the (vector) factors of a vector product and, in case of dyadics, the discrimination between antecedent and consequent.

Thus, if $\dot{\mathbf{r}}$ means the "differential coefficient" or the flux of $\mathbf{r}$ with respect to any scalar variable x, *i.e.* if $\dot{\mathbf{r}}$ stands for $\frac{\partial \mathbf{r}}{\partial x}$, the flux of a sum $\mathbf{a}+\mathbf{b}$ of two vectors is $\dot{\mathbf{a}}+\dot{\mathbf{b}}$, and that of the scalar product $\mathbf{ab}$ is $\mathbf{a}\dot{\mathbf{b}}+\dot{\mathbf{a}}\mathbf{b}$. Similarly, for the vector product,

$$\frac{\partial}{\partial x}\mathrm{V}\mathbf{ab} = \mathrm{V}\dot{\mathbf{a}}\mathbf{b} + \mathrm{V}\mathbf{a}\dot{\mathbf{b}},$$

and so on. Or, using "differentials,"

$$d(\mathbf{a}+\mathbf{b}) = d\mathbf{a} + d\mathbf{b}, \quad d(\mathbf{ab}) = \mathbf{a}\,d\mathbf{b} + \mathbf{b}\,d\mathbf{a},$$

where $d\mathbf{a}$, $d\mathbf{b}$ are infinitesimal vectors. Similarly

$$d\mathrm{V}\mathbf{ab} = \mathrm{V}\mathbf{a}\,d\mathbf{b} - \mathrm{V}\mathbf{b}\,d\mathbf{a},$$

the second term being negatived to compensate for the reversal of the order of its two factors. Further explanations, if at all needed, will be found in any treatise on Vector Analysis.*

Such being the case, it will be enough briefly to consider the passage of a pencil of rays $\mathbf{r}$ through a single refracting surface (reflection being, formally, but a sub-case). Thus, returning to the fundamental refraction formula (II), Section 1, we have for any ray of an incident pencil, let us say for the "central" ray of the pencil,

$$\mathbf{s}' = \mathbf{s} + g\mathbf{n}, \tag{II}$$

where $\mathbf{n}$ is the normal of the refracting surface, of any shape (free of singularities), at the point struck by the central incident ray $\mathbf{s}$. Let the refractive indices μ, μ' be fixed, and let us pass from the central ray $\mathbf{s}$ (to which corresponds the central ray $\mathbf{s}'$ of the refracted pencil) to an infinitesimally neighbouring ray $\mathbf{s}+d\mathbf{s}$. If all the rays of the pencil are parallel, then, of course, $d\mathbf{s}=0$, $\mathbf{s}$ representing the whole pencil. But in general we shall have $d\mathbf{s}\neq 0$. At any rate, since $\mathbf{s}^2=\mu^2$, the relation $\mathbf{s}\,d\mathbf{s}=0$, or $d\mathbf{s}\perp\mathbf{s}$, will hold. The corresponding refracted ray will be $\mathbf{s}'+d\mathbf{s}'$, and $d\mathbf{s}'\perp\mathbf{s}'$ as before. Now, passing from one ray to another means, in general, varying the normal $\mathbf{n}$, and therefore also g. Thus, subtracting the original or central $\mathbf{s}'$ from the new vector $\mathbf{s}'+d\mathbf{s}'$, we have for the required change of the refracted ray,

$$d\mathbf{s}' = d\mathbf{s} + g\,d\mathbf{n} + \mathbf{n}\,dg, \tag{43}$$

where, by the definition of g, p. 2,

$$dg = \mu \sin i \,.\, di - \mu' \sin i' \,.\, di',$$

and by the sine-law, $\mu' \cos i' \,.\, di' = \mu \cos i \,.\, di$, so that ultimately

$$dg = f(i)\, di,$$

f being a known function of i alone. Notice that, $\mathbf{n}$ being a unit vector, we have always $\mathbf{n}\,d\mathbf{n}=0$. In the case of reflection it is enough to replace μ' by $-\mu$.

In essence, formula (43) already contains the answer to all thinkable questions concerning the properties of the infinitesimal refracted pencil in terms of those of the incident one. It is enough to read it intelligently and to develop it appropriately in each particular case.

* Cf., for instance, pp. 22-48 of my *Vectorial Mechanics*. Only a small part of what is there given will be required for the purposes of geometrical optics.

Let u, v be any curvilinear coordinates defining the position of a point at the refracting surface σ. Then "incident pencil given" means $d\mathbf{s}$ given as a function of u, v throughout an infinitesimal region round u_0, v_0, the point struck by the central ray. Now, the surface σ being given (round that place at least), say, through the expression of its squared line-element,

$$dl^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{44}$$

$d\mathbf{n}$ is also known as a function of the two coordinates. The same is true of dg, which differs from di by a known factor $f(i)$.

Thus (43) finds $d\mathbf{s}'$ as function of u, v round u_0, v_0; that is, it gives all the properties of the refracted pencil. For $\mathbf{s}'$, $\mathbf{n}$ or $\mathbf{s}'$, u_0, v_0 give the direction and the position of the central pencil, and

$$\mathbf{s}' + d\mathbf{s}' \text{ with } u_0 + du, \quad v_0 + dv$$

localize and determine the direction of any ray of the refracted pencil. Thus every question concerning the latter pencil is reduced to purely geometrical constructions.

Employing u, v (for instance, ϕ, θ for a sphere) as our independent variables, we have, of course,

$$dg = f(i)\left(\frac{\partial i}{\partial u}du + \frac{\partial i}{\partial v}dv\right),$$

and in quite the same way

$$d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u}du + \frac{\partial \mathbf{n}}{\partial v}dv.$$

If the incident pencil consists, for instance, of parallel rays, so that $d\mathbf{s} = 0$, then

$$d\mathbf{s}' = g\,d\mathbf{n} + \mathbf{n}\,dg, \tag{45}$$

whence $\mathbf{n}\,d\mathbf{s}' = dg$, and $\mathrm{V}\mathbf{n}\,d\mathbf{s}' = g\mathrm{V}\mathbf{n}\,d\mathbf{n}$, two convenient formulae replacing (45). The chief question in any such investigation is whether the neighbour ray passing through $P(u_0 + du, v_0 + dv)$ meets at all, at finite or infinite distance, the central refracted ray passing through $P_0(u_0, v_0)$. In order to put this question in vectorial language, introduce the infinitesimal vector $d\mathbf{l}$, drawn on the surface from P_0 to P, its size dl being determined by the quadratic form (44) with E, F, G as given functions of u, v. Then the said rays meet or not (are coplanar or not) according as the scalar magnitude

$$Q = d\mathbf{l}\,\mathrm{V}\mathbf{s}'(\mathbf{s}' + d\mathbf{s}')$$

vanishes or not. Remembering that $\mathrm{V}\mathbf{s}'\mathbf{s}'=0$, identically, this criterion-expression is reduced to

$$Q=d\mathbf{l}\,\mathrm{V}\mathbf{s}'d\mathbf{s}',$$

which, by (II) and (45), is

$$Q=[dg\,\mathrm{V}\mathbf{sn}+g\mathrm{V}\mathbf{s}\,d\mathbf{n}+g^2\mathrm{V}\mathbf{n}\,d\mathbf{n}]d\mathbf{l}. \tag{46}$$

Given the refracting surface and the incident pencil $\mathbf{s}$, together with the central normal $\mathbf{n}$, this expression will vanish or not according to the choice of the point P, *i.e.* of the vectorial line element $d\mathbf{l}$. In general we shall have $Q=0$ only for certain particular directions of $d\mathbf{l}$ upon the interface. Having established the conditions for $Q=0$, it will be a comparatively easy task to localize the point of intersection of the P-ray with the central ray of the refracted pencil, and thus to build up the caustics. If the incident beam is normal, then, whatever the shape of the refracting surface, the most convenient curvilinear coordinates u, v will be those measured along the lines of curvature of that surface drawn through P_0 as origin, in which case (44) is reduced to $dl^2=E\,du^2+G\,dv^2$. For oblique incidence, however, another network may in certain cases be found more convenient.

As an exercise the reader may treat the case of *normal* incidence upon *any* refracting surface, such as an ellipsoid, a cylinder or a cone; and the case of *oblique* incidence for a *spherical* surface, thus deriving vectorially the well-known formulae for the sagittal and the meridional section of a pencil.

Since these and similar problems have nothing to do with "ray-tracing," the proper argument of this little volume, we cannot enter upon them any further. The last mentioned, and similar, exercises may, however, be warmly recommended to the reader as good and interesting opportunities of training in the use of vector language.

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